THE CONSTRUCTION OF QUASI-INVARIANT MEASURES

BY

YITZHAK KATZNELSON AND BENJAMIN WEISS

ABSTRACT

If a homeomorphism possesses a non-atomic ergodic measure it has recurrent points. If it has recurrent points then there are uncountably many inequivalent ergodic quasi-invariant measures.

There has recently been some interest expressed in the construction of many inequivalent quasi-invariant ergodic measures for uniquely ergodic systems. M. Keane, in [1], has given an explicit construction of uncountably many inequivalent such measures for the irrational rotations of the circle while W. Krieger, [2], has applied category methods to obtain such collections for any uniquely ergodic homeomorphism. We propose to describe a simple construction of such families of measures under quite general conditions, that are in fact necessary. For completeness sake we review the definitions. We will be dealing with a fixed compact metric space X and a homeomorphism ϕ of X onto itself. All measures will be finite measures defined on the Borel field $\mathscr{B}$ of X. A measure μ is said to be *ergodic* if

(1) $$E \in \mathscr{B},\ \phi E = E \text{ imply } \mu(E)\,\mu(X \backslash E) = 0.$$

It is said to be a *quasi-invariant* measure (q.i.) if

(2) $$\mu(E) = 0 \text{ iff } \mu(\phi E) = 0.$$

Since any atomic measure whose support is the orbit of a single point is an ergodic q.i. we promptly lose interest in such measures, and concentrate on finding *non-atomic* ergodic q.i.'s. The following is a necessary condition for the existence of non-atomic ergodic measures.

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LEMMA 1. *If μ is a non-atomic ergodic measure then $\mu-$ a.e. point of X is recurrent* (i.e. $\phi^n x$ returns infinitely often to any deleted neighborhood of x).

PROOF. Let $A_\varepsilon = \{x : \inf_{n \neq 0} d(\phi^n x, x) \geqq \varepsilon\}$, where d is a metric on X. Since μ – a.e. point is non periodic (ergodicity), it suffices to show that $\mu(A_\varepsilon) = 0$. Suppose that $\mu(A_\varepsilon) > 0$, then there is an $A \subset A_\varepsilon$ with diam $(A) \leqq \varepsilon/3$ and $\mu(A) > 0$. Observe that, for $n \neq 0$, $\phi^n A \cap A = \varnothing$. Since μ is non-atomic there is a $B \subset A$ with $\mu(B)\ \mu(A \setminus B) > 0$. But then $\bigcup_n \phi^n B = E$ is invariant and disjoint from $A \setminus B$, thus $\mu(E)\ \mu(X \setminus E) > 0$ contradicting the ergodicity. □

The existence of some recurrent point is all that we shall need for our construction of *uncountably many* inequivalent ergodic q.i.'s. The next simple observation means that we can forget about the quasi-invariant part.

Given a measure μ define $\phi^n\mu$ by $\phi^n\mu(E) = \mu(\phi^n E)$. Let $c_n > 0$, with $\sum_{-\infty}^{\infty} c_n = 1$, then

$$\tilde{\mu} = \sum_{-\infty}^{\infty} c_n \phi^n \mu \tag{3}$$

is a q.i. and indeed the "minimal" q.i. with respect to which μ is absolutely continuous.

For later reference note that if μ is ergodic so is $\tilde{\mu}$.

LEMMA 2. *Let $\{\mu_\alpha\}$ be an uncountable collection of mutually singular non-atomic measures. Then $\{\tilde{\mu}_\alpha\}$ contains an uncountable family of mutually singular measures.*

PROOF. The lemma clearly follows if we show that for $\mu \in \{\mu_\alpha\}$ there are at most countably many $\nu \in \{\mu_\alpha\}$ such that $\tilde{\nu}$, $\tilde{\mu}$ are not mutually singular. This in turn follows if we show that for each integer n, there are at most countably many ν's which are not singular with respect to $\phi^n\mu$. If ν is not singular with respect to $\phi^n\mu$ write $\nu = \nu_c + \nu_s$ where $\nu_c \neq 0$ is the absolutely continuous $(\phi^n\mu)$, i.e., $\nu_c \in L^1(\phi^n\mu)$. Since the ν's are mutually singular so are the ν_c's which means that they have disjoint supports (which are sets of positive $\phi^n\mu$ measure) and their number is therefore finite or countable. □

One final preliminary lemma is well known, a proof is included for the reader's convenience.

LEMMA 3. *Suppose that for arbitrarily large N there are disjoint sets $A_1^{(N)}, A_2^{(N)}, \cdots, A_N^{(N)}$ such that $\mu(X \setminus \cup A_i^{(N)}) = 0$ and for any i,j there is an $n = n(i,j)$ and $c = c(i,j) > 0$ such that $\phi^n A_i^{(N)} = A_j^{(N)}$ and for $E \subset A_i^{(N)}$, $\mu(E)$*

$= c \cdot \mu(\phi^n E)$. *If in addition the collections* $\{A_i^{(N)}\}_{i \leqq N}$ *generates* $\mathscr{B}$, *then* μ *is an ergodic measure for* ϕ.

PROOF. Since $\{A_i^{(N)}\}_{i \leqq N}$ generates $\mathscr{B}$, if $\phi E = E$ and $\mu(E)\,\mu(X \setminus E) > 0$, for any $\varepsilon > 0$ there is an $A_i^{(N)}$ with $\mu(E \cap A_i^{(N)}) \geqq (1-\varepsilon)\,\mu(A_i^{(N)})$. Since $\phi^n E = E$ it follows that $\mu(E \cap A_j^{(N)}) \geqq (1-\varepsilon)\,\mu(A_j^{(N)})$ for all j, i.e. $\mu(E) \geqq (1-\varepsilon)\,\mu(X)$ contradicting $(X \setminus E) > 0$. □

THEOREM. *A necessary and sufficient condition for a homeomorphism* ϕ *of a compact metric space* X, *to have uncountably many inequivalent non-atomic ergodic q.i.'s is that* ϕ *have a recurrent point.*

PROOF. The necessity is contained in Lemma 1. Suppose then that x_0 is a recurrent point. Since $\phi x_0 \neq x_0$ there is a compact neighborhood $C(0)$ of x_0 such that $\phi C(0) \cap C(0) = \varnothing$, set $\phi C(0) = C(1)$. By shrinking $C(0)$ if necessary ensure that max (diameter $C(0)$, diameter $C(1)) \leqq 1$.

By the recurrence of x_0 there is an n with $x_0 \neq \phi^n x_0 \in \operatorname{Int} C(0)$. Let $C(00)$ be a compact neighborhood of x_0 such that

$$C(01) = \phi^n C(00) \subset \operatorname{Int}(C(0))$$

and

$$C(01) \cap C(00) = \varnothing.$$

Define

$$C(1i) = \phi C(0i) \quad (i = 0, 1)$$

and by shrinking $C(00)$ if necessary ensure that

$$\max_{0 \leqq i,j \leqq 1} \operatorname{diam}(C(ij)) \leqq \tfrac{1}{2}$$

By continuing in this fashion we construct sets $C(\varepsilon_1\varepsilon_2 \cdots \varepsilon_k)$ $\varepsilon_i = 0, 1$ that are mutually homeomorphic, for fixed k, under powers of ϕ, and such that the intersection

$$\bigcap_k C(\varepsilon_1\varepsilon_2 \cdots \varepsilon_k) = \pi(\varepsilon_1, \varepsilon_2, \cdots \varepsilon_n, \cdots)$$

consists of a single point for each sequence of zeros and ones. Indeed π establishes a homeomorphism between $Y = \prod_1^\infty \{0, 1\}$ with the product topology and

$$C = \bigcap_k \bigcup_{\varepsilon_i = 0,1} C(\varepsilon_1\varepsilon_2 \cdots \varepsilon_k)$$

To define a measure on X that is concentrated on C it suffices to define a measure on Y and use π to map it onto $C \subset X$. Let μ_p be the product measure on Y determined by $\mu_p(0) = p$, $\mu_p(1) = 1 - p$. For distinct $p \in (0, 1)$ these measures are mutually singular—since, as is well known, they are *invariant* ergodic measures for the unilateral shift. The μ_p are non-atomic and thus Lemma 2 completes the proof. □

We've written the construction for the group of integers acting on a compact space. Any countable group would have served as well. Needless to say, uniquely ergodic homeomorphisms with a non-atomic invariant measure satisfy the hypotheses of Lemma 2 and thus we recover the Keane-Krieger result.

REFERENCES

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HEBREW UNIVERSITY OF JERUSALEM
AND
WEIZMANN INSTITUTE OF SCIENCE, REHOVOT